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Global existence for rough solutions of a fourth-order nonlinear wave equation

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In this paper, we prove that the cubic fourth-order wave equation is globally well-posed in $H^s(\mathbb{R}^n)$ for $s > \min\{\frac{n-2}{2}, \frac{n}{4}\}$ by following the Bourgain's Fourier truncation idea in Bourgain (1998) [2]. To avoid some troubles, we technically make use of the Strichartz estimate for low frequency part and high frequency part, respectively. As far as we know, this is the first result on the low regularity behavior of the fourth-order wave equation.

1. Introduction

This paper is concerned with the global well-posedness in $H^s(\mathbb{R}^n)$ of the Cauchy problem for the following defocusing cubic fourth-order wave equation

$$\begin{cases} \partial_{tt}u + \Delta^2 u + u + |u|^2 u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = u_0, \qquad \partial_t u|_{t=0} = u_1, \end{cases} \tag{1.1}$$

with rough initial data $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$ for $0 < s < 2$.

Recently, the Cauchy problem (1.1) with initial data $(u_0, u_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ has been extensively studied. For instance, the well-posedness and the energy scattering theory of this Cauchy problem have been studied by Pausader [14,16]. Furthermore Pausader and Strauss have proved the analyticity of the scattering operator in [15]. Meanwhile, one can see the result on analyticity of the scattering operator for semilinear dispersive equations in [4].

Many authors [3,6,7,9–11,19] have studied the local well-posedness (as well as global well-posedness) in fractional Sobolev spaces for the Cauchy problem of general semilinear wave and Schrödinger equations under minimal regularity assumptions on the initial data. For example, Tao [19] established the sharp local well-posedness of a nonlinear wave equation. Kenig, Ponce, and Vega [7] obtained the global well-posedness of nonlinear wave equations with rough initial data (in particular, in $L^4(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)$, $\frac{3}{4} < s < 1$ for cubic wave equation). They utilized the Fourier truncation method discovered by Bourgain [2] in their proof. And also D. Fang, C. Miao, and B. Zhang [10] extended Kenig–Ponce–Vega's result to $n \geqslant 4$. Recently, I. Gallagher and F. Planchon [6] presented an alternative approach to obtain the same result as [7]. H. Bahouri and J.-Y. Chemin [1] further proved the global well-posedness for $s = \frac{3}{4}$ by using a nonlinear interpolation method and logarithmic estimates from S. Klainerman and D. Tataru [8]. Besides, Roy [17] obtained the global well-posedness for cubic wave equation in H^s, $\frac{13}{18} < s < 1$, by using the I-method [5] and scaling transformation. However, if we applies the I-method to

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this fourth-order wave equation, we will possibly meet a problem caused by the lack of the scaling property. As mentioned in [18], the author while considering the low regularity of the Klein–Gordon equation, points out that a more delicate analysis is needed to overcome the lack of scaling property, if one tries to use the I-method. More studies and discussions on the low regularity of nonlinear wave or dispersive Schrödinger equations could be found in [3,12,13,20]. However, very little seems to be known about the global existence theory for the fourth-order wave equation with rough initial data. Now we present our main result:

Theorem 1.1. *Let* $\min\{\frac{n-2}{2}, \frac{n}{4}\} < s < 2$. *If* $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$ *with* $3 \leqslant n \leqslant 7$, *then the defocusing cubic fourth-order wave equation* (1.1) *is globally well-posed in* $H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$.

Remark 1.1. (i) We remark that the case of $n = 3$ for the theorem is similar to the case of $n = 4$, one can modify the proof to obtain $s > \frac{n-2}{2}$. The index $\frac{n-2}{2}$ comes from the similar argument of $n = 4$, since $\frac{n}{4} = \frac{n-2}{2}$ when $n = 4$.

(ii) When $n = 7$, by substituting $L^2_{\Delta T} L^{r^*}$ with $L^2_{\Delta T} L^{14}$, we can modify the proof of the case of $n = 5, 6$ to gain an analogous theorem, which holds for $s > \frac{7}{4}$ and $n = 7$.

(iii) The regularity condition $s > \frac{n}{4}$ heavily depends on our techniques. Our ideal perfect result is $s \geqslant \max\{0, \frac{n-4}{2}\}$. To this end, bilinear estimate, interaction Morawetz estimate and other techniques have been further extended in [2,5] could be employed in our future studies. However, the achievement of that ideal perfect result cannot be that easy. We hope to return to the problem of proving sharp or better results for the fourth order wave equation in a future work.

2. Preliminaries

In this section, we introduce some notations and definitions that will be frequently used in this paper. If X, Y are nonnegative quantities, we occasionally use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \leqslant CY$ for some constant C, which may depend on $\|u_0\|_{H^s}$ and $\|u_1\|_{H^{s-2}}$. Pairs of conjugate indices are written as p and p' with $1 \leqslant p \leqslant \infty$ and $1/p + 1/p' = 1$. And also we set $2^+ = 2 + \varepsilon$ for any small $\varepsilon > 0$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

We will also need the Littlewood–Paley projection operators. Specifically, let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \leqslant 2$, which equals 1 on the ball $|\xi| \leqslant 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$, we define the Littlewood–Paley operators

$$\widehat{P_{\leqslant N} f}(\xi) := \varphi\left(\frac{\xi}{N}\right)\hat{f}(\xi), \qquad \widehat{P_{>N} f}(\xi) := \left(1 - \varphi\left(\frac{\xi}{N}\right)\right)\hat{f}(\xi),$$
$$\widehat{P_N f}(\xi) := \left(\varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right)\right)\hat{f}(\xi).$$

Similarly we can define $P_{<N}$, $P_{\geqslant N}$, and $P_{M<\cdot\leqslant N} = P_{\leqslant N} - P_{\leqslant M}$, whenever M and N are dyadic numbers. The Littlewood–Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every L^p_x and $\dot{H}^s_x$ space for $1 \leqslant p \leqslant \infty$ and $s \geqslant 0$, moreover, they also obey the following Bernstein estimates

$$\|P_{\geqslant N} f\|_{L^p} \lesssim N^{-s} \big\||\nabla|^s P_{\geqslant N} f\big\|_{L^p}, \qquad \big\||\nabla|^s P_{\leqslant N} f\big\|_{L^p} \lesssim N^s \|P_{\leqslant N} f\|_{L^p},$$
$$\big\||\nabla|^{\pm s} P_N f\big\|_{L^p} \sim N^{\pm s} \|P_N f\|_{L^p}, \qquad \|P_{\leqslant N} f\|_{L^q} \lesssim N^{\frac{n}{p} - \frac{n}{q}} \|P_{\leqslant N} f\|_{L^p},$$
$$\|P_N f\|_{L^q} \lesssim N^{\frac{n}{p} - \frac{n}{q}} \|P_N f\|_{L^p},$$

where $s \geqslant 0$ and $1 \leqslant p \leqslant q \leqslant \infty$.

We go on this section by giving definition of a "half-wave" operator T_t. This "half-wave" operator is defined by

$$T_t f(x) = \exp\big(it\sqrt{1 + \Delta^2}\big) f(x), \quad \text{for } f \in L^1 + L^2,$$

which is equivalent to

$$\mathcal{F}(T_t f)(\xi) = \exp\big(it\sqrt{1 + |\xi|^4}\big)\mathcal{F}(f)(\xi)$$

for $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We recall the Strichartz-type estimate in [14] for the operator T_t, but here we would like to rewrite this type of Strichartz's estimates for low frequency part and high frequency part, respectively.

The Strichartz estimates involve the following admissible definitions:

Definition 2.1. A pair of Lebesgue space exponents (q,r) is called a $\frac{n}{4}$-admissible pair, or denoted by $(q,r)\in\Lambda_L$, when $q,r\geqslant 2$, $(q,r,\frac{n}{4})\neq(2,\infty,1)$ satisfying

$$\frac{4}{q}\leqslant n\left(\frac{1}{2}-\frac{1}{r}\right). \tag{2.1}$$

Definition 2.2. A pair of Lebesgue space exponents (q,r) is called sharp $\frac{n}{2}$-admissible pair, or denoted by $(q,r)\in\Lambda_H$, when $q,r\geqslant 2$, $(q,r,\frac{n}{2})\neq(2,\infty,1)$ satisfying

$$\frac{2}{q}=n\left(\frac{1}{2}-\frac{1}{r}\right). \tag{2.2}$$

We conclude this section by giving the following Strichartz type estimates whose proofs can be found in [14].

Lemma 2.1 *(Strichartz's estimate for low frequency). (See [14].) For $u_0\in L^2(\mathbb{R}^n)$ and $F\in L^{q_1'}(\mathbb{R};L^{r_1'}(\mathbb{R}^n))$, we have the following Strichartz type estimates for the half-wave operator T_t:*

$$\begin{aligned}
&\|P_{\leqslant 1}T_t u_0\|_{L^{q_0}(I;L^{r_0}(\mathbb{R}^n))}\leqslant C\|u_0\|_{L^2},\\
&\left\|\int_{\mathbb{R}} P_{\leqslant 1}T_{-\tau}F(x,\tau)\,d\tau\right\|_{L^2(\mathbb{R}^n)}\leqslant C\|F\|_{L^{q_1'}(\mathbb{R};L^{r_1'}(\mathbb{R}^n))},\\
&\left\|\int_0^t P_{\leqslant 1}T_{t-\tau}F(x,\tau)\,d\tau\right\|_{L^{q_0}(I;L^{r_0}(\mathbb{R}^n))}\leqslant C\|F\|_{L^{q_1'}(\mathbb{R};L^{r_1'}(\mathbb{R}^n))},
\end{aligned} \tag{2.3}$$

where (q_0,r_0) and (q_1,r_1) belong to Λ_L.

Lemma 2.2 *(Strichartz's estimate for high frequency). (See [14].) For $u_0\in L^2(\mathbb{R}^n)$ and $F\in L^{q_1'}(\mathbb{R};L^{r_1'}(\mathbb{R}^n))$, we have the following Strichartz type estimates for the half-wave operator T_t:*

$$\begin{aligned}
&\|P_{>1}T_t u_0\|_{L^{q_0}(I;L^{r_0}(\mathbb{R}^n))}\leqslant C\|u_0\|_{L^2},\\
&\left\|\int_{\mathbb{R}} P_{>1}T_{-\tau}F(x,\tau)\,d\tau\right\|_{L^2(\mathbb{R}^n)}\leqslant C\|F\|_{L^{q_1'}(\mathbb{R};L^{r_1'}(\mathbb{R}^n))},\\
&\left\|\int_0^t P_{>1}T_{t-\tau}F(x,\tau)\,d\tau\right\|_{L^{q_0}(I;L^{r_0}(\mathbb{R}^n))}\leqslant C\|F\|_{L^{q_1'}(\mathbb{R};L^{r_1'}(\mathbb{R}^n))},
\end{aligned} \tag{2.4}$$

where (q_0,r_0) and (q_1,r_1) belong to Λ_H.

We remark at the end of this section that solutions of (1.1), at least formally, satisfy the energy conservation law

$$\int_{\mathbb{R}^n}\frac{1}{2}\left(|\partial_t u|^2+|u|^2+|\Delta u|^2+\frac{1}{2}|u|^4\right)dx=\text{constant}. \tag{2.5}$$

3. Proof of the main theorem

Our proof relies on the Bourgain's Fourier truncation idea in [2]. Roughly speaking, one splits the data into two pieces: high and low frequencies. The latter solves the original problem in a time interval $[0,\Delta T]$ with ΔT depending on the regularity of the original data. Since the latter only has low frequencies, its solution, called $v(t)$, has enough regularity, so it satisfies the energy conservation law. One uses $v(t)$ to find $w(t)=u(t)-v(t)$ in the time interval $[0,\Delta T]$. We observe that the inhomogeneous part $z(t)$ of $w(t)$ is in H^2. Thus, we add $z(\Delta T)$ to $v(\Delta T)$, and repeat the arguments for the involved norms' growth. This has to be taken into account to make the process uniform. It is here where the restriction on $s>\min\{\frac{n-2}{2},\frac{n}{4}\}$ appears.

We split the initial data $(u_0,u_1)\in H^s\times H^{s-2}$ into two parts

$$u_0=u_0^l+u_0^h,\qquad u_1=u_1^l+u_1^h,$$

where

$$u_0^l = P_{\leqslant N}u_0, \qquad u_1^l = P_{\leqslant N}u_1,$$

with large N to be chosen. Therefore, we can see that

$$\|u_0^l\|_{H^\sigma} + \|u_1^l\|_{H^{\sigma-2}} \leqslant C(\|u_0\|_{H^s}, \|u_1\|_{H^{s-2}})N^{\sigma-s}, \quad \text{with } \sigma \geqslant s, \tag{3.1}$$

and

$$\|u_0^h\|_{H^{\tilde\sigma}} + \|u_1^h\|_{H^{\tilde\sigma-2}} \leqslant C(\|u_0\|_{H^s}, \|u_1\|_{H^{s-2}})N^{\tilde\sigma-s}, \quad \text{for } \tilde\sigma \in [0,s]. \tag{3.2}$$

We only focus on the proof of Theorem 1.1 holds for the dimension $n = 4, 5, 6$. Recalling the remark below Theorem 1.1, one can prove the other cases can be proved by slightly modifying the following steps.

Step 1. We consider the fourth-order wave equation with regular data

$$\begin{cases} \partial_{tt}v + \Delta^2 v + v + |v|^2 v = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^n, \\ v|_{t=0} = u_0^l, \qquad \partial_t u|_{t=0} = u_1^l, \end{cases} \tag{3.3}$$

and this equation is equivalent to the following integral equation

$$v(t) = \frac{T_t + T_{-t}}{2}u_0^l + \frac{T_t - T_{-t}}{2i\omega}u_1^l - \int_0^t \frac{T_{t-\tau} - T_{\tau-t}}{2i\omega}|v|^2 v(\tau)\,d\tau \tag{3.4}$$

with $\omega = (1+\Delta^2)^{1/2}$. The solution $v(t,x)$ enjoys the energy conservation law (2.5). Thus, it follows from (3.1) that

$$\|(\partial_t v, \sqrt{1+\Delta^2})v(t)\|_{L^2} + \|v(t)\|_{L^4}^2 \sim N^{2-s}, \quad t \geqslant 0. \tag{3.5}$$

Now we solve this equation in the time interval $[0, \Delta T]$ with $\Delta T = N^{-\alpha}$ where

$$\alpha = \begin{cases} 2, & 2 > s \geqslant \frac{n-2}{2}, \\ 1, & \frac{n}{4} < s < \frac{n-2}{2}. \end{cases}$$

The motivation of the choice of such length of time will be clear at the end of this proof. Note $2 > s > \min\{\frac{n-2}{2}, \frac{n}{4}\}$, it follows that

$$\frac{4(2-s)}{8-n} - \alpha \leqslant 0 \tag{3.6}$$

and

$$-\frac{\alpha}{3} + 2\left(-s + \frac{n-2}{3}\right) \leqslant 0. \tag{3.7}$$

Let $r = 2^+$ and $r^* = \frac{2r}{r-2}$, then a simple triangle inequality yields

$$\|v\|_{L^2_{\Delta T}L^{r^*}} \leqslant I_1 + I_2 + I_3, \tag{3.8}$$

where

$$I_1 = \left\|\frac{T_t + T_{-t}}{2}u_0^l\right\|_{L^2_{\Delta T}L^{r^*}}, \qquad I_2 = \left\|\frac{T_t - T_{-t}}{2i\omega}u_1^l\right\|_{L^2_{\Delta T}L^{r^*}}, \qquad I_3 = \left\|\int_0^t \frac{T_{t-\tau} - T_{\tau-t}}{2i\omega}|v|^2 v(\tau)\,d\tau\right\|_{L^2_{\Delta T}L^{r^*}}.$$

We firstly break I_1 into two pieces

$$I_1 \leqslant \|P_{\geqslant 1}T_t u_0^l\|_{L^2_{\Delta T}L^{r^*}} + \|P_{<1}T_t u_0^l\|_{L^2_{\Delta T}L^{r^*}}.$$

Then in the case of $n = 4$, the Hölder inequality yields

$$\begin{aligned} I_1 &\leqslant \|P_{\geqslant 1}\langle\nabla\rangle^{(n-2)/2-s}T_t\langle\nabla\rangle^s u_0^l\|_{L^2_{\Delta T}L^{\frac{2n}{n-2}}} + \Delta T^{\frac{1}{2}-\frac{1}{r}}\|P_{<1}\langle\nabla\rangle^{-s}T_t\langle\nabla\rangle^s u_0^l\|_{L^r_{\Delta T}L^{r^*}} \\ &\leqslant \|P_{\geqslant 1}T_t\langle\nabla\rangle^s u_0^l\|_{L^2_{\Delta T}L^{\frac{2n}{n-2}}} + \Delta T^{\frac{1}{2}-\frac{1}{r}}\|P_{<1}T_t\langle\nabla\rangle^s u_0^l\|_{L^r_{\Delta T}L^{r^*}}. \end{aligned}$$

And in the case of $5 \leqslant n \leqslant 6$, we apply the Sobolev embedding and Bernstein's inequality to high frequency part and low frequency part, respectively, to show

$$I_1 \leqslant \left\| P_{\geqslant 1} T_t \langle\nabla\rangle^{(n-2)/2} u_0^l \right\|_{L^2_{\Delta T} L^{\frac{2n}{n-2}}} + \left\| P_{<1} T_t \langle\nabla\rangle^{(n-4)/2} u_0^l \right\|_{L^2_{\Delta T} L^{\frac{2n}{n-4}}}.$$

We can rewrite the above one as

$$\begin{aligned} I_1 &\leqslant \left\| P_{\geqslant 1} \langle\nabla\rangle^{(n-2)/2-s} T_t \langle\nabla\rangle^s u_0^l \right\|_{L^2_{\Delta T} L^{\frac{2n}{n-2}}} + \left\| P_{<1} \langle\nabla\rangle^{(n-4)/2-s} T_t \langle\nabla\rangle^s u_0^l \right\|_{L^2_{\Delta T} L^{\frac{2n}{n-4}}} \\ &\leqslant \max\left\{1, N^{(n-2)/2-s}\right\} \left\| P_{\geqslant 1} T_t \langle\nabla\rangle^s u_0^l \right\|_{L^2_{\Delta T} L^{\frac{2n}{n-2}}} + \left\| P_{<1} T_t \langle\nabla\rangle^s u_0^l \right\|_{L^2_{\Delta T} L^{\frac{2n}{n-4}}}. \end{aligned}$$

It follows from the Strichartz estimates (2.3) and (2.4) that

$$I_1 \leqslant \max\left\{1, N^{(n-2)/2-s}\right\} \left(\left\| \langle\nabla\rangle^s u_0^l \right\|_{L^2} + \left\| \langle\nabla\rangle^s u_0^l \right\|_{L^2} \right) \lesssim \max\left\{1, N^{(n-2)/2-s}\right\}. \tag{3.9}$$

As a consequence of the similar argument for I_2, we also have

$$I_2 \lesssim \max\left\{1, N^{(n-2)/2-s}\right\}. \tag{3.10}$$

Again we decompose the third term into two parts as follows

$$I_3 \leqslant \left\| P_{\geqslant 1} \int_0^t \frac{T_{t-\tau}}{2i\omega} |v|^2 v(\tau)\, d\tau \right\|_{L^2_{\Delta T} L^{r*}} + \left\| P_{<1} \int_0^t \frac{T_{t-\tau}}{2i\omega} |v|^2 v(\tau)\, d\tau \right\|_{L^2_{\Delta T} L^{r*}},$$

and a similar argument as before and the energy conservation lead to

$$\begin{aligned} I_3 &\lesssim \left\| P_{\geqslant 1} \langle\nabla\rangle^{(n-2)/2-2} \left(v^3\right) \right\|_{L^1_{\Delta T} L^2} + \left\| P_{<1} \langle\nabla\rangle^{(n-4)/2-2} \left(v^3\right) \right\|_{L^1_{\Delta T} L^2} \\ &\leqslant \Delta T \|v\|^3_{L^\infty(L^n)} \leqslant \Delta T \|v\|^{\frac{6(6-n)}{8-n}}_{L^\infty(L^4)} \|v\|^{\frac{3(n-4)}{8-n}}_{L^\infty(L^{\frac{2n}{n-4}})} \leqslant \Delta T N^{\frac{6}{8-n}(2-s)}. \end{aligned} \tag{3.11}$$

Collecting (3.8)–(3.11), we obtain that

$$\|v\|_{L^2_{\Delta T} L^{r*}} \lesssim 1 + N^{(n-2)/2-s} + \Delta T N^{\frac{6}{8-n}(2-s)}. \tag{3.12}$$

Step 2. In this step, we consider the Cauchy problem for $w(t) = u(t) - v(t)$ with initial data (u_0^h, u_1^h),

$$\begin{cases} \partial_{tt} w + \Delta^2 w + w = -|w+v|^2(w+v) + |v|^2 v & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ w|_{t=0} = u_0^h, \qquad \partial_t w|_{t=0} = u_1^h, \end{cases} \tag{3.13}$$

and its integral equation is

$$w(t) = \frac{T_t + T_{-t}}{2} u_0^h + \frac{T_t - T_{-t}}{2i\omega} u_1^h - \int_0^t \frac{T_{t-\tau} - T_{\tau-t}}{2i\omega} F(\tau)\, d\tau := \frac{T_t + T_{-t}}{2} u_0^h + \frac{T_t - T_{-t}}{2i\omega} u_1^h + z(t), \tag{3.14}$$

where $F(t) = |w+v|^2(w+v) - |v|^2 v$. It is clear that

$$\left|F(t)\right| \lesssim |w|^3 + |w||v|^2 := F_1 + F_2. \tag{3.15}$$

To estimate $\|w\|_{L^3_{\Delta T} L^6}$, it follows from the integral equation (3.14) that

$$\|w\|_{L^3_{\Delta T} L^6} \leqslant II_1 + II_2 + II_3 + II_4, \tag{3.16}$$

where

$$II_1 = \left\| \frac{T_t + T_{-t}}{2} u_0^h \right\|_{L^3_{\Delta T} L^6}, \qquad II_2 = \left\| \frac{T_t - T_{-t}}{2i\omega} u_1^h \right\|_{L^3_{\Delta T} L^6},$$

$$II_3 = \left\| P_{\geqslant 1} \int_0^t \frac{T_{t-\tau} - T_{\tau-t}}{2i\omega} F(\tau)\, d\tau \right\|_{L^3_{\Delta T} L^6}, \qquad II_4 = \left\| P_{<1} \int_0^t \frac{T_{t-\tau} - T_{\tau-t}}{2i\omega} F(\tau)\, d\tau \right\|_{L^3_{\Delta T} L^6}.$$

Now we establish the estimates of the four terms step by step. The bound of the first two terms are easy to obtain. Since $1 \ll N$, thus we can write

$$II_1 \lesssim \left\| P_{\geqslant 1} T_t u_0^h \right\|_{L^3_{\Delta T} L^6} \leqslant \left\| P_{\geqslant 1} T_t \langle \nabla \rangle^{(n-2)/3} u_0^h \right\|_{L^3_{\Delta T} L^{\frac{6n}{3n-4}}},$$

and the Strichartz estimate and (3.2) together with $s \geqslant (n-2)/3$ give

$$II_1 \lesssim \left\| \langle \nabla \rangle^{(n-2)/3} u_0^h \right\|_{L^2} \leqslant N^{-s+(n-2)/3}. \tag{3.17}$$

A similar argument yields that

$$II_2 \leqslant N^{-s+(n-2)/3}. \tag{3.18}$$

Now we turn to estimate the last two terms. For the third term, we have that, by Sobolev's embedding as before

$$II_3 \leqslant \left\| P_{\geqslant 1} \int_0^t \frac{T_{t-\tau}}{2i\omega} F(\tau)\, d\tau \right\|_{L^3_{\Delta T} L^6} \leqslant \left\| P_{\geqslant 1} \int_0^t T_{t-\tau} \langle \nabla \rangle^{-2+(n-2)/3} F(\tau)\, d\tau \right\|_{L^3_{\Delta T} L^{\frac{6n}{3n-4}}}.$$

Splitting the F into two parts, again the Strichartz estimate implies

$$II_3 \leqslant \left\| \langle \nabla \rangle^{-2+(n-2)/3} F_1 \right\|_{L^1_{\Delta T} L^2} + \left\| \langle \nabla \rangle^{-2+(n-2)/3} F_2 \right\|_{L^1_{\Delta T} L^2}.$$

On one hand, the Sobolev embedding and Hölder inequality yield

$$\left\| \langle \nabla \rangle^{-2+(n-2)/3} F_1 \right\|_{L^1_{\Delta T} L^2} \leqslant \left\| w^3 \right\|_{L^1_{\Delta T} L^{\frac{6n}{16+n}}} \leqslant \|w\|^{\frac{n-2}{2}}_{L^3_{\Delta T} L^6} \|w\|^{\frac{8-n}{2}}_{L^3_{\Delta T} L^{\frac{6n}{n+4}}} \leqslant \Delta T^{\frac{8-n}{6}} \|w\|^{\frac{n-2}{2}}_{L^3_{\Delta T} L^6} \|w\|^{\frac{8-n}{2}}_{L^\infty H^{\frac{n-2}{3}}}. \tag{3.19}$$

On the other hand, we also have that

$$\left\| \langle \nabla \rangle^{-2+(n-2)/3} F_2 \right\|_{L^1_{\Delta T} L^2} \leqslant \left\| w v^2 \right\|_{L^1_{\Delta T} L^{\frac{6n}{16+n}}} \leqslant \Delta T \|w\|_{L^\infty L^{\frac{6n}{n+4}}} \|v\|^2_{L^\infty L^n} \leqslant \Delta T N^{\frac{4(2-s)}{8-n}} \|w\|_{L^\infty H^{\frac{n-2}{3}}}. \tag{3.20}$$

Combining (3.19) and (3.20), one can see that

$$II_3 \lesssim \Delta T^{\frac{8-n}{6}} \|w\|^3_{L^3_{\Delta T} L^6 \cap L^\infty H^{\frac{n-2}{3}}} + \Delta T N^{\frac{4(2-s)}{8-n}} \|w\|_{L^\infty H^{\frac{n-2}{3}}}. \tag{3.21}$$

An analogous argument with a slight modification helps us to control the fourth term. We start this estimate at

$$II_4 \leqslant \left\| P_{<1} \int_0^t \frac{T_{t-\tau}}{2i\omega} F(\tau)\, d\tau \right\|_{L^3_{\Delta T} L^6} \leqslant \left\| P_{<1} \int_0^t T_{t-\tau} \langle \nabla \rangle^{-2+(n-4)/3} F(\tau)\, d\tau \right\|_{L^3_{\Delta T} L^{\frac{6n}{3n-8}}}.$$

The Strichartz estimate for low frequencies shows that

$$II_4 \lesssim \left\| \langle \nabla \rangle^{-2+(n-4)/3} F_1 \right\|_{L^1_{\Delta T} L^2} + \left\| \langle \nabla \rangle^{-2+(n-4)/3} F_2 \right\|_{L^1_{\Delta T} L^2}. \tag{3.22}$$

And again the Sobolev embedding and Hölder's inequality give

$$\left\| \langle \nabla \rangle^{-2+(n-4)/3} F_1 \right\|_{L^1_{\Delta T} L^2} \leqslant \left\| w^3 \right\|_{L^1_{\Delta T} L^{\frac{6n}{20+n}}} \leqslant \|w\|^{\frac{n+2}{4}}_{L^3_{\Delta T} L^6} \|w\|^{\frac{10-n}{4}}_{L^3_{\Delta T} L^{\frac{6n}{n+8}}} \leqslant \Delta T^{\frac{10-n}{12}} \|w\|^{\frac{n+2}{4}}_{L^3_{\Delta T} L^6} \|w\|^{\frac{10-n}{4}}_{L^\infty H^{\frac{n-4}{3}}},$$

and

$$\left\| \langle \nabla \rangle^{-2+(n-4)/3} F_2 \right\|_{L^1_{\Delta T} L^2} \leqslant \left\| w v^2 \right\|_{L^1_{\Delta T} L^{\frac{6n}{20+n}}} \leqslant \Delta T \|w\|_{L^\infty L^{\frac{6n}{n+8}}} \|v\|^2_{L^\infty L^n} \leqslant \Delta T N^{\frac{4(2-s)}{8-n}} \|w\|_{L^\infty H^{\frac{n-4}{3}}}.$$

Making a combination of the above two, we obviously have

$$II_4 \leqslant \Delta T^{\frac{10-n}{12}} \|w\|^3_{L^3_{\Delta T} L^6 \cap L^\infty H^{\frac{n-4}{3}}} + \Delta T N^{\frac{4(2-s)}{8-n}} \|w\|_{L^\infty H^{\frac{n-4}{3}}} \leqslant \Delta T^{\frac{10-n}{12}} \|w\|^3_{L^3_{\Delta T} L^6 \cap L^\infty H^{\frac{n-2}{3}}} + \Delta T N^{\frac{4(2-s)}{8-n}} \|w\|_{L^\infty H^{\frac{n-2}{3}}}. \tag{3.23}$$

Finally collecting (3.16)–(3.18), (3.21) and (3.23), we obtain

$$\|w\|_{L^3_{\Delta T} L^6} \lesssim N^{\frac{n-2}{3}-s} + \Delta T N^{\frac{4(2-s)}{8-n}} \|w\|_{L^\infty H^{\frac{n-2}{3}}} + \left(\Delta T^{\frac{8-n}{6}} + \Delta T^{\frac{10-n}{12}} \right) \|w\|^3_{L^3_{\Delta T} L^6 \cap L^\infty H^{\frac{n-2}{3}}}. \tag{3.24}$$

To control $\|w\|_{L^3_{\Delta T}L^6}$, we need to estimate the $\|w\|_{L^\infty H^{\frac{n-2}{3}}}$ by a desirable bound. Now we repeat the above process and give its bound. To estimate $\|w\|_{L^\infty H^{\frac{n-2}{3}}}$, we utilize the integral equation (3.14) to write that

$$\|w\|_{L^\infty H^{\frac{n-2}{3}}} \leqslant III_1 + III_2 + III_3 + III_4,$$

where III_1, III_2, III_3 and III_4 are analogues to II_1, II_2, II_3, II_4 in (3.16), with a modification on the norm. For the first three terms, we can bound them by making use of a similar argument as before. Before estimating the fourth term, we remark that

$$\|f\|_{H^{\frac{n-2}{3}}} \leqslant \|f\|_{H^{\frac{n-4}{3}}}$$

if $\hat{f}$ is supported in the ball $B(0,1)$. Therefore, we can replace the norm $L^\infty H^{\frac{n-2}{3}}$ by $L^\infty H^{\frac{n-4}{3}}$ for the fourth term, which only has low frequencies. Thus we can proceed this estimate as before to obtain

$$\|w\|_{L^\infty H^{\frac{n-2}{3}}} \lesssim N^{\frac{n-2}{3}-s} + \Delta T N^{\frac{4(2-s)}{8-n}} \|w\|_{L^\infty H^{\frac{n-2}{3}}} + \left(\Delta T^{\frac{8-n}{6}} + \Delta T^{\frac{10-n}{12}}\right) \|w\|^3_{L^3_{\Delta T}L^6 \cap L^\infty H^{\frac{n-2}{3}}}. \tag{3.25}$$

A combination of (3.6), (3.24) and (3.25), the standard continuous argument yields

$$\|w\|_{L^3_{\Delta T}L^6 \cap L^\infty H^{\frac{n-2}{3}}} \lesssim N^{-s+(n-2)/3}, \tag{3.26}$$

by choosing large N enough.

Although the above one gives a bound of $\|w\|_{L^\infty(L^2)}$, we need a better estimate for the lower derivatives of w. Again it follows from the integral equation (3.14) and the Strichartz estimates that

$$\|w\|_{L^\infty_{\Delta T}L^2} \leqslant \|u_0^h\|_{L^2} + \|u_1^h\|_{H^{-2}} + IV_1 + IV_2, \tag{3.27}$$

where

$$IV_1 = \left\| P_{\geqslant 1} \int_0^t \frac{T_{t-\tau} - T_{\tau-t}}{2i\omega} F(\tau)\, d\tau \right\|_{L^\infty_{\Delta T}L^2}, \qquad IV_2 = \left\| P_{<1} \int_0^t \frac{T_{t-\tau} - T_{\tau-t}}{2i\omega} F(\tau)\, d\tau \right\|_{L^\infty_{\Delta T}L^2}.$$

Hence, by the decomposition of F and Strichartz estimate, we have

$$IV_1 \leqslant \left\| P_{\geqslant 1} \frac{|\nabla|^{\frac{n}{3}}}{\sqrt{1+\Delta^2}} |\nabla|^{-\frac{n}{3}} F_1 \right\|_{L^1_{\Delta T}L^2} + \left\| P_{\geqslant 1} \frac{1}{\sqrt{1+\Delta^2}} F_2 \right\|_{L^1_{\Delta T}L^2},$$

and further the Sobolev inequality shows, if $n=4$,

$$IV_1 \leqslant \left\| P_{\geqslant 1} \frac{|\nabla|^{\frac{n}{3}}}{\sqrt{1+\Delta^2}} F_1 \right\|_{L^1_{\Delta T}L^{\frac{6}{5}}} + \|P_{\geqslant 1} F_2\|_{L^1_{\Delta T}L^{1+}}$$

and if $n=5,6$,

$$IV_1 \leqslant \left\| P_{\geqslant 1} \frac{|\nabla|^{\frac{n}{3}}}{\sqrt{1+\Delta^2}} F_1 \right\|_{L^1_{\Delta T}L^{\frac{6}{5}}} + \|P_{\geqslant 1} F_2\|_{L^1_{\Delta T}L^{\frac{2n}{n+4}}}.$$

By dropping the negative derivatives on the high frequency, we get the bound of the first term, if $n=4$,

$$IV_1 \leqslant \|w^3\|_{L^1_{\Delta T}L^{\frac{6}{5}}} + \|wv^2\|_{L^1_{\Delta T}L^{1+}} \leqslant \Delta T^{\frac{1}{3}} \|w\|_{L^\infty_{\Delta T}L^2} \|w\|^2_{L^3_{\Delta T}L^6} + \Delta T \|w\|_{L^\infty_{\Delta T}L^2} \|v\|^2_{L^\infty_{\Delta T}L^{4+}} \tag{3.28}$$

and if $n=5,6$,

$$IV_1 \leqslant \|w^3\|_{L^1_{\Delta T}L^{\frac{6}{5}}} + \|wv^2\|_{L^1_{\Delta T}L^{\frac{2n}{n+4}}} \leqslant \Delta T^{\frac{1}{3}} \|w\|_{L^\infty_{\Delta T}L^2} \|w\|^2_{L^3_{\Delta T}L^6} + \Delta T \|w\|_{L^\infty_{\Delta T}L^2} \|v\|^2_{L^\infty_{\Delta T}L^n}. \tag{3.29}$$

For the second term, by discarding of the negative inhomogeneous derivatives on low frequency, the Bernstein inequality yields that

$$IV_2 \leqslant \left\| P_{<1} \frac{1}{\sqrt{1+\Delta^2}} F_1 \right\|_{L^1_{\Delta T}L^2} + \left\| P_{<1} \frac{1}{\sqrt{1+\Delta^2}} F_2 \right\|_{L^1_{\Delta T}L^2} \leqslant \|P_{<1}F_1\|_{L^1_{\Delta T}L^{\frac{6}{5}}} + \|P_{<1}F_2\|_{L^1_{\Delta T}L^{\frac{2n}{n+4}}}.$$

Thus, we still have

$$IV_2 \lesssim \Delta T^{\frac{1}{3}} \|w\|_{L^\infty_{\Delta T} L^2} \|w\|^2_{L^3_{\Delta T} L^6} + \Delta T \|w\|_{L^\infty_{\Delta T} L^2} \|v\|^2_{L^\infty_{\Delta T} L^n}.$$

Observe that (3.6) and $\|w\|_{L^3_{\Delta T} L^6} \lesssim N^{-s+(n-2)/3}$, it follows from (3.27) that

$$\|w\|_{L^\infty_{\Delta T} L^2} \lesssim N^{-s} + N^{-\frac{\alpha}{3}-2s+\frac{2(n-2)}{3}} \|w\|_{L^\infty_{\Delta T} L^2} + c\|w\|_{L^\infty_{\Delta T} L^2}.$$

Note that (3.7) and choose c small sufficiently, we obtain with large N,

$$\|w\|_{L^\infty_{\Delta T} L^2} \lesssim N^{-s}. \tag{3.30}$$

By interpolation (3.26) and (3.30), we have for any $0 \leqslant \epsilon \leqslant \frac{n-2}{3}$,

$$\|w\|_{L^\infty_{\Delta T} H^\epsilon} \lesssim N^{\epsilon-s}.$$

On the other hand, one can see that there exists small ϵ such that

$$\|w\|_{L^\infty_{\Delta T} L^{\tilde r}} \leqslant \|w\|_{L^\infty_{\Delta T} H^\epsilon} \lesssim N^{\epsilon-s} \tag{3.31}$$

for $\tilde r = 2^+$.

Step 3. We shall estimate $\|(\partial_t z, \langle\Delta\rangle z)(\Delta T)\|_{L^2_x} + \|z(t)\|^2_{L^4}$, with $z(t)$ defined in (3.14). By the definition of $z(t)$, we have

$$\left\|\big(\partial_t z, \langle\Delta\rangle z\big)(\Delta T)\right\|_{L^2_x} \lesssim \left\| \int_0^t T_{t-\tau} F(\tau)\, d\tau \right\|_{L^\infty_{\Delta T} L^2}.$$

Note that the admissible pair $(\infty, 2)$ belongs not only to Λ_L but also to Λ_H, it is not necessary to split the right hand into low frequency and high frequency. And so we can directly apply the Strichartz estimate to it and control the right hand by

$$\|F\|_{L^1_{\Delta T} L^2} \lesssim \|w\|^3_{L^3_{\Delta T} L^6} + \left\|w v^2\right\|_{L^1_{\Delta T} L^2}.$$

On one hand, from (3.26) one has that

$$\|w\|^3_{L^3_{\Delta T} L^6} \lesssim N^{-3s+(n-2)}. \tag{3.32}$$

On the other hand, recalling $r = 2^+$ and $r^* = \frac{2r}{r-2}$, it follows from (3.12) and (3.31) that there exists a small $\epsilon > 0$ such that

$$\left\|w v^2\right\|_{L^1_{\Delta T} L^2} \lesssim \|w\|_{L^\infty_{\Delta T} L^{\frac{2r}{4-r}}} \|v\|^2_{L^2_{\Delta T} L^{r^*}} \lesssim N^{\epsilon-s}\big(1 + N^{\frac{n-2}{2}-s} + N^{-\alpha} N^{\frac{6(2-s)}{8-n}}\big)^2. \tag{3.33}$$

Hence, (3.32) and (3.33) imply that

$$\left\|\big(\partial_t z(t), \langle\Delta\rangle z(t)\big)(\Delta T)\right\|_{L^2_x} \lesssim N^{\epsilon-s}\big(1 + N^{\frac{n-2}{2}-s} + N^{-\alpha} N^{\frac{6(2-s)}{8-n}}\big)^2. \tag{3.34}$$

Now we turn to estimate the potential energy $\|z(t)\|^2_{L^4_x}$. By the definition of $z(t)$ again, we have by Minkowski's inequality and Sobolev's embedding

$$\left\|z(\Delta T)\right\|_{L^4_x} \lesssim \int_0^{\Delta T} \left\| \frac{\sin\omega(\Delta T-\tau)}{\omega} F(\tau) \right\|_{L^4_x} d\tau \lesssim \left\|F(t)\right\|_{L^1_{\Delta T} L_x^{\frac{4n}{n+8}}}. \tag{3.35}$$

To bound the last expression we use (3.5), (3.26) and (3.30) to write

$$\left\||w|^2 w\right\|_{L^1_{\Delta T} L_x^{\frac{4n}{n+8}}} = \|w\|^3_{L^3_{\Delta T} L_x^{\frac{12n}{n+8}}} \leqslant \Delta T^{\frac{8-n}{4n}} \|w\|^{\frac{3(5n-8)}{4n}}_{L^3_{\Delta T} L^6_x} \|w\|^{\frac{3(8-n)}{4n}}_{L^\infty_{\Delta T} L^2_x} \lesssim \Delta T^{\frac{8-n}{4n}} N^{\frac{(n-2-3s)(5n-8)}{4n}} N^{\frac{-3s(8-n)}{4n}}, \tag{3.36}$$

and for the case $n = 4$,

$$\left\|v^2 w\right\|_{L^1_{\Delta T} L_x^{\frac{4n}{n+8}}} \leqslant \|w\|_{L^1_{\Delta T} L^2_x} \|v\|^2_{L^\infty_{\Delta T} L_x^{\frac{4n}{8-n}}} \leqslant \Delta T \|w\|_{L^\infty_{\Delta T} L^2_x} N^{\frac{3}{2}(2-s)} \leqslant N^{\frac{1}{2}(n-2-3s)}, \tag{3.37}$$

and for the case $5 \leqslant n \leqslant 6$,

$$\begin{aligned}\|v^2w\|_{L^1_{\Delta T}L_x^{\frac{4n}{n+8}}} &\leqslant \|w\|_{L^1_{\Delta T}L_x^{\frac{4n}{3(8-n)}}}\|v\|^2_{L^\infty_{\Delta T}L_x^{\frac{2n}{n-4}}} \leqslant (\Delta T)^{\frac{24-n}{4n}}\|w\|^{\frac{72-11n}{4n}}_{L^\infty_{\Delta T}L^2_x}\|w\|^{\frac{15n-72}{4n}}_{L^3_{\Delta T}L^6_x}\|v\|^2_{L^\infty_{\Delta T}L_x^{\frac{2n}{n-4}}}\\ &\leqslant N^{-\frac{24-n}{4n}}N^{\frac{-s(72-11n)}{4n}}N^{\frac{(n-2-3s)(5n-24)}{4n}}N^{2(2-s)} \leqslant N^{\frac{1}{2}(n-2-3s)}.\end{aligned} \tag{3.38}$$

Therefore, a collection of (3.35)–(3.38) gives that

$$\|z(\Delta T)\|^2_{L^4_x} \lesssim N^{n-2-3s}, \tag{3.39}$$

which together with (3.34) yields

$$\big\|\big(\partial_t z(t), \langle\Delta\rangle z(t)\big)(\Delta T)\big\|_{L^2_x} + \|z(\Delta T)\|^2_{L^4_x} \lesssim N^{\epsilon-s}\big(1+N^{\frac{n-2}{2}-s}+N^{-\alpha}N^{\frac{6(2-s)}{8-n}}\big)^2. \tag{3.40}$$

Step 4. We solve the Cauchy problem (3.3) in the time interval $[\Delta T, 2\Delta T]$ with data

$$\big(v(\Delta T)+z(\Delta T), \partial_t v(\Delta T)+\partial_t z(\Delta T)\big) \tag{3.41}$$

and then we insert its solution $v(t)$ in (3.13) to solve this Cauchy problem in the time interval $[\Delta T, 2\Delta T]$ with data

$$\left(\cos\omega(\Delta T)u_0^h + \frac{\sin\omega(\Delta T)}{\omega}u_1^h, -\sin\omega(\Delta T)\omega u_0^h + \cos\omega(\Delta T)u_1^h\right) \tag{3.42}$$

and repeat the above estimates in Steps 1, 2 and 3.

For arbitrary large T which we want to reach at, we can repeat the above argument

$$\frac{T}{\Delta T} = TN^\alpha \tag{3.43}$$

times. Then, the total added to the expression in the right-hand side of (3.5) will be

$$CTN^\alpha N^{\epsilon-s}\big(1+N^{\frac{n-2}{2}-s}+N^{-\alpha}N^{\frac{6(2-s)}{8-n}}\big)^2.$$

To make the above computations uniformly, we just need that

$$CTN^\alpha N^{\epsilon-s}\big(1+N^{\frac{n-2}{2}-s}+N^{-\alpha}N^{\frac{6(2-s)}{8-n}}\big)^2 + CN^{2-s} \leqslant 2CN^{2-s}. \tag{3.44}$$

If $2 > s \geqslant \frac{n-2}{2}$, then we can choose $\alpha = 2$ to meet the requirements of (3.6) and (3.7), and ensure (3.44). If $s \leqslant \frac{n-2}{2}$, to guarantee (3.44), it reduces to consider the following linear optimal problem: under the restriction (3.6), (3.7) and

$$\begin{cases}\alpha - s + (n-2) - 2s < 2 - s,\\ \alpha - s - 2\alpha + \dfrac{12(2-s)}{8-n} < 2 - s,\end{cases}$$

find the value of α that minimizes the low boundedness of s. Through drawing the figure of these restrictions, we can solve this optimal problem by choosing $\alpha = 1$ to ensure $s > \frac{n}{4}$.

In conclusion, if $\frac{n-2}{2} \geqslant s > \frac{n}{4}$, the inequality (3.44) holds by fixing the value of ϵ sufficiently close to 0 and $\alpha = 1$. And in the case of $2 > s > \frac{n-2}{2}$, the inequality (3.44) still holds by fixing the value of ϵ sufficiently close to 0 and $\alpha = 2$. Taking N sufficiently large, we complete the proof of our main theorem.

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